# Optimally handling commitment issues in online throughput maximization

Franziska Eberle* Nicole Megow* Kevin Schewior†

December 23, 2019

**Abstract**

We consider a fundamental online scheduling problem in which jobs with processing times and deadlines arrive online over time at their release dates. The task is to determine a feasible preemptive schedule on a single machine that maximizes the number of jobs that complete before their deadline. Due to strong impossibility results for competitive analysis, it is commonly required that jobs contain some *slack* $\varepsilon > 0$, which means that the feasible time window for scheduling a job is at least $1+\varepsilon$ times its processing time. In this paper, we resolve the question on how to handle commitment requirements which enforce that a scheduler has to guarantee at a certain point in time the completion of admitted jobs. This is very relevant, e.g., in providing cloud-computing services and disallows last-minute rejections of critical tasks. We give an algorithm with an optimal competitive ratio of $\Theta(1/\varepsilon)$ for the online throughput maximization problem when a scheduler must commit upon starting a job. Somewhat surprisingly, this is the same optimal performance bound (up to constants) as for scheduling without commitment. If commitment decisions must be made before a job's slack becomes less than a $\delta$-fraction of its size, we prove a competitive ratio of $\mathcal{O}(\varepsilon/((\varepsilon-\delta)\delta))$ for $0 < \delta < \varepsilon$. This result interpolates between commitment upon starting a job and commitment upon arrival. For the latter commitment model, it is known that no (randomized) online algorithms does admit any bounded competitive ratio.

## 1 Introduction

We consider the following fundamental online scheduling model: jobs from an unknown job set arrive online over time at their *release dates* $r_j$. Each job has a *processing time* $p_j \geq 0$ and a *deadline* $d_j$. There is a single machine to process these jobs or a subset of them. A job is said to *complete* if it receives $p_j$ units of processing time within the interval $[r_j, d_j)$. We allow *preemption*, i.e., the processing of a job can be interrupted at any time. In a feasible schedule, no two jobs are ever processing at the same time. The number of completed jobs in a feasible schedule is called *throughput*. The task is to find a feasible schedule with maximum throughput.

*Department for Mathematics and Computer Science, University of Bremen, Germany. Email: {feberle,nicole.megow}@uni-bremen.de. Partially supported by the German Science Foundation (DFG) under contract ME 3825/1.

†Departement of Mathematics and Computer Science, University of Cologne, Germany. Email: kschewior@gmail.com. Partially supported by the DAAD within the PRIME program using funds of BMBF and the EU Marie Curie Actions.

As jobs arrive online, we cannot hope to find an optimal schedule. To assess the performance of online algorithms, we resort to standard *competitive analysis*. This means, we compare the throughput of an online algorithm with the throughput achievable by an optimal offline algorithm that knows the job set in advance.

It is well-known that "tight" jobs with $d_j - r_j \approx p_j$ prohibit competitive online decision making as jobs must start immediately and do not leave a chance for observing online arrivals. Thus, it is commonly required that jobs contain some *slack* $\varepsilon > 0$, i.e., every job $j$ satisfies $d_j - r_j \geq (1+\varepsilon)p_j$. The competitive ratio of our online algorithm will be a function of $\varepsilon$; the greater the slack, the better should the performance of our algorithm be. This slackness parameter has been considered in previous work, e.g., in [2, 4, 8, 11, 12, 18, 20]. Other results for scheduling with deadlines use speed scaling, which can be viewed as another way to add slack to the schedule, e.g., [1, 3, 13, 19].

In this paper, we focus on the question how to handle *commitment* requirements in online throughput maximization. Modeling commitment addresses the issue that a good-throughput schedule may abort jobs close to their deadlines in favor of many shorter and more urgent tasks [10], which may not be acceptable for the job owner. Consider a company that starts outsourcing mission-critical processes to external clouds and that needs a guarantee that jobs complete before a certain time point when they cannot be moved to another computing cluster anymore. In other situations, a commitment to complete jobs might be required even earlier just before starting the job, e.g., for a faultless copy of a database [8].

Different commitment models have been formalized [2, 8, 18]. The requirement to commit at a job's release date has been ruled out for online throughput maximization by strong impossibility results [8]. We distinguish *(i) commitment upon job admission* and *(ii) $\delta$-commitment*. In the first model, an algorithm may discard a job any time before its start, we say its admission. This reflects a situation such as the faultless copy of a database. In the second model, $\delta$-commitment, an online algorithm must commit to complete a job when its remaining slack is not less than a $\delta$-fraction of the job size, for $0 < \delta < \varepsilon$. The latest time for committing to job $j$ is then $d_j - (1+\delta)p_j$. This models an early enough commitment (parameterized by $\delta$) for mission-critical jobs. These models have been studied and, recently, a first unified approach has been presented in [8]. Gaps in the performance bounds remained and it was left open if scheduling with commitment is even "harder" than without commitment.

In this work, we close these gaps for online throughput maximization on a single machine and answer the "hardness" question to the negative. We give an algorithmic framework that achieves the provably best competitive ratio (up to constants) for the aforementioned commitment models. Somewhat surprisingly, we show that the same competitive ratio of $\mathcal{O}(\frac{1}{\varepsilon})$ can be achieved for both, scheduling *without* commitment and *with* commitment upon admission. Informally speaking, we show that scheduling with commitment is not harder than scheduling without.

## 1.1 Previous results

Preemptive online scheduling with hard deadlines as well as models for admission control have been studied rigorously, see, e.g., [5, 11, 12] and the references therein. Already in the 90s several impossibility results were shown for jobs without slack [6, 7, 15–17]. The only positive result independent of slack for online throughput maximization without commitment seems to be an $\mathcal{O}(1)$-competitive algorithm using randomization [14]. For instances with $\varepsilon$-

slack and no commitment requirement, we gave a best possible $\mathcal{O}(1/\varepsilon)$-competitive algorithm with a matching lower bound [8].

Throughput maximization with commitment has attracted researchers more recently [2, 8, 18]. We summarize the state-of-the art for the particular problem of online throughput maximization with commitment on a single machine. We presented in our recent work [8] a universal algorithmic framework, called region algorithm, which achieved bounded competitive ratios for several commitment models and even the tight result for scheduling without commitment. More precisely, the region algorithm is $\mathcal{O}(1/\varepsilon^2)$-competitive for commitment upon admission and $\mathcal{O}(\varepsilon/((\varepsilon-\delta)\delta^2))$-competitive, for $0 < \delta < \varepsilon$, in the $\delta$-commitment model. This improves on an earlier algorithm by Azar et al. [2] for the *$\delta$-commitment model* (in the context of truthful mechanisms for a weighted setting) that is $\mathcal{O}(\frac{1}{\varepsilon^2})$-competitive if the slack $\varepsilon$ is sufficiently large. We also showed a lower bound of $\Omega(\frac{1}{\varepsilon})$ for scheduling without commitment, which is tight in that model and clearly holds also for the more restrictive commitment models. A significant gap between lower and upper bounds remained.

In a natural generalization of our problem, jobs have associated individual weights and we aim for a schedule with maximum weighted throughput. The special case with each job $j$ satisfying $w_j = p_j$ (aka machine utilization) is well understood. A simple greedy algorithm achieves the best possible competitive ratio $\Theta(1/\varepsilon)$ [9, 11] in both commitment models, even for commitment upon arrival. In this setting, even scheduling with commitment on parallel machines is tractable [20]. It is worth mentioning that machine utilization without commitment even allows for constant competitive ratios independent of slack [6, 15, 16, 21]. General weighted (and even unweighted) throughput maximization is much less tractable. It has been shown that the general weighted problem is hopeless under commitment requirements and no bounded competitive ratio is possible in any of the aforementioned models [2, 8, 18].

### 1.2 Our results and techniques

Our main result is one optimal algorithm for online throughput maximization with commitment. When a scheduler must commit upon starting a job, we show a competitive ratio of $\mathcal{O}(1/\varepsilon)$. This is best possible as there is a lower bound of $\Omega(1/\varepsilon)$ for online throughput maximization even without commitment [8]. In the $\delta$-commitment model, where commitment decisions must be made before a job's slack becomes less than a $\delta$-fraction of its size, we prove an upper bound of $\mathcal{O}(\varepsilon/((\varepsilon-\delta)\delta))$ for $0 < \delta < \varepsilon$. This result interpolates nicely between both "extreme models", namely commitment upon starting a job and commitment upon arrival. For small $\delta$, say $\delta < \varepsilon/2$, the competitive ratio is $\Theta(1/\varepsilon)$ which is the best one can hope for, even without commitment. For large $\delta$, with $\delta \to \epsilon$, the commitment requirement tightens such that commitment decisions must be made essentially upon job arrival, and the competitive ratio is unbounded (even for randomized algorithms) [8].

The challenge in online scheduling with commitment is that, once the algorithm committed to the completion of a job, the remaining slack of this job has to be spent very carefully. The key of our algorithm is a job admission scheme which is implemented by different parameters. The three high-level objectives are: (i) we never start a job for the first time if its remaining slack is too small (parameter $\delta$), (ii) during the processing of an admitted job, we admit only significantly shorter jobs (parameter $\gamma$), and (iii), for each admitted shorter job, we avoid admitting too many other jobs of similar size (parameter $\beta$). While the first two goals are quite natural and have been implemented in some ways before [8, 18], the third goal is crucial for our new and tight result.

The intuition is the following: suppose we committed to complete a job with processing time 1 and have only a slack of $\mathcal{O}(\varepsilon)$ left before the deadline of this job. Suppose that $c$ substantially smaller jobs of size $1/c$ arrive where $c$ is the competitive ratio we aim for. On the one hand, if we do not accept any of them, we cannot hope to achieve $c$-competitiveness. On the other hand, accepting too many of them fills up the slack and, thus, leaves no room for even smaller jobs. The idea is to keep the flexibility for future small jobs by grouping jobs of similar size into classes and accepting only one out of $\mathcal{O}(1/\varepsilon)$ jobs per class, which is enough for a $\mathcal{O}(1/\varepsilon)$-approximation. This is implemented by carefully defining a blocking period (after which our algorithm is named) that follows a successfully scheduled job and in which no job of similar or larger processing time is accepted.

The analysis splits into two parts: first, we show that the blocking algorithm completes all admitted jobs on time, and second, we show that the blocking algorithm admits sufficiently many jobs to be competitive. As a key contribution on the technical side, we prove a strong technical lemma concerning the processing volume any feasible solution can achieve compared to the volume of our online algorithm that was used in a weaker form in earlier work. As a side result, we can substantially shorten the analysis of an earlier algorithm in [8].

# 2 An optimal algorithm for commitment

In this section we describe the *blocking algorithm* which handles scheduling with commitment. We assume that the slackness constant $\varepsilon > 0$ and, in the $\delta$-commitment model, $0 < \delta < \varepsilon$ is given. If $\delta$ is not part of the input or if $\delta \leq \varepsilon/2$, we set $\delta = \frac{\varepsilon}{2}$. Moreover we state the main result on the blocking algorithm.

## 2.1 The blocking algorithm

Our algorithm commits to completing any job when it has started processing for the first time, we say the job has been *admitted.* When a job is admitted, its remaining slack has to be spent very carefully. Thus, the algorithm transfers the admission decision to the shortest admitted and not yet completed job. Then, a job only admits significantly shorter jobs and prevents the admission of too many jobs of similar size. To this end, we maintain two types of intervals for each admitted job, a *scheduling interval* and a *blocking period.* A job can only be processed in its scheduling interval and, thus, it has to complete in this interval while admitting other jobs. Job $j$ only admits jobs that are smaller by a factor of $\gamma = \frac{\delta}{16} < 1$. For an admitted job $i$, job $j$ creates a blocking period of length at most $\beta p_i$, where $\beta = \frac{16}{\delta}$, which blocks the admission of similar-length jobs. These intervals are shown in Figure 1.

For **scheduling**, the algorithm follows the simple *Shortest Processing Time (SPT)* order which is independent of the admission scheme. SPT guarantees that a job $j$ has highest priority in the blocking periods of any job $i$ admitted by $j$.

For **admitting** jobs, the algorithm keeps track of *available jobs* at any time point $t$. A job $j$ with $r_j \leq t$ is called available if it has not yet been admitted by the algorithm and its deadline is not too close, i.e., $d_j - t \geq (1+\delta)p_j$.

Whenever a job $j$ is released or available at a time that is not contained in the scheduling interval of any other job, the shortest such job $j$ is admitted immediately, creating the scheduling interval $S(j) = [r_j, r_j + (1+\delta)p_j) := [a_j, e_j)$ and an empty blocking period $B(j) = \emptyset$. In general, however, the blocking period is a finite union of time intervals associated with job $j$, and its size is the sum of lengths of the intervals, denoted by $|B(j)|$. Three events can trigger

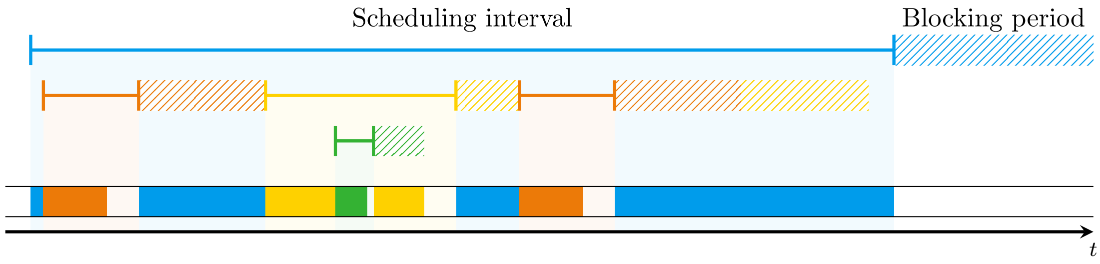


**Figure 1:** Scheduling interval, blocking period, and processing intervals.

a decision of the algorithm at time $t$: the release of a job, the end of a blocking period, and the end of a scheduling interval. In any of these three cases, the algorithm calls the class admission routine. This subroutine checks if $j$, the shortest job whose scheduling interval contains $t$, can admit the currently shortest available job $i$.

To this end, any admitted job $j$ classifies available jobs $i$ with $r_i \in S(j)$ and $p_i < \gamma p_j$ depending on their processing time. More precisely, job $j$ maintains a *class structure* $(\mathcal{C}_c(j))_{c\in\mathbb{N}_0}$ where $\mathcal{C}_c(j)$ contains all jobs $i$ with $r_i \in S(j)$ and $\frac{\gamma}{2^{c+1}} p_j \le p_i < \frac{\gamma}{2^c} p_j$. Only jobs $i \in \mathcal{C}_c(j)$ for $c \in \mathbb{N}_0$ qualify for admission by $j$. Upon admission by $j$, job $i$ obtains two disjoint consecutive intervals, the *scheduling interval* $S(i) = [a_i, e_i)$ and the *blocking period* $B(i)$ of size at most $\beta p_i$. At the admission of job $i$, the blocking period $B(i)$ is planned to start at $e_i$, the end of $i$'s scheduling interval. During $B(i)$ of job $i \in \mathcal{C}_c(j)$, $j$ only admits jobs $i'$ of *higher* classes, i.e., $i' \in \mathcal{C}_{c'}(j)$ for $c' > c$. Particularly, $j$ only admits job $i \in \mathcal{C}_c(j)$ if the blocking period of the last admitted job in $\mathcal{C}_c(j)$ has completed.

Hence, when job $j$ decides if it admits the currently shortest available job $i$ at time $t$, it makes sure that $i$ indeed belongs to a class $\mathcal{C}_c(j)$ and that no higher class $c' \ge c$ is blocking $t$, i.e., it checks that $t \notin B(i')$ for a job $i' \in \mathcal{C}_{c'}(j)$. In this case, we say that $i$ is a *child* of $j$ and that $j$ is the *parent* of $i$, denoted by $\pi(i) = j$. If job $i$ is admitted at time $t$ by job $j$, the algorithm sets $a_i = t$ and $e_i = a_i + (1+\delta)p_i$ and assigns the scheduling interval $S(i) = [a_i, e_i)$ to $i$.

If $e_i \le e_j$, the routine sets $f_i = \min\{e_j, e_i + \beta p_i\}$ which implies $B(i) = [e_i, f_i)$. As the scheduling and blocking periods of the children $k$ of $j$ are supposed to be disjoint, we have to **update the blocking periods**. First consider the job $k \in \mathcal{C}_{c'}(j)$ for $c' < c$ whose blocking period contains $t$ and let $[e'_k, f'_k)$ be the maximal interval of $B(k)$ containing $t$. We set $f''_k = \min\{e_j, f'_k + (1+\delta+\beta)p_i\}$ and replace the interval $[e'_k, f'_k)$ by $[e'_k, t) \cup [t + (1+\delta+\beta)p_i, f''_k)$. For all other jobs $k \in \mathcal{C}_{c'}(j)$ with $B(k) \cap [t, \infty) \ne \emptyset$, we replace the remaining part of their blocking period $[e'_k, f'_k)$ by $[e'_k + (1+\delta+\beta)p_i, f''_k)$ where $f''_k := \min\{e_j, f'_k + (1+\delta+\beta)p_i\})$. In this update we follow the convention $[e, f) = \emptyset$ if $f \le e$. Observe that the length of the blocking period might decrease due to such updates.

Note that $e_i > e_j$ is also possible as $j$ does not take the end of its own scheduling interval $e_j$ into account when admitting jobs. Thus, the scheduling interval of $i$ would end outside $j$'s scheduling interval and inside $j$'s blocking period. During $B(j)$, $\pi(j)$, the parent of $j$, did not allocate the interval $[e_j, e_i)$ for completing jobs admitted by $j$ but for ensuring its own completion. Hence, the completion of both $i$ and $\pi(j)$ is not necessarily guaranteed anymore. To prevent this, we **modify all scheduling intervals** $S(k)$ (including $S(j)$) that contain time $t$ and the corresponding blocking periods $B(k)$. For each admitted job $k$ with $t \in S(k)$ (i.e., including $j$) and $e_i > e_k$ we set $e_k = e_i$. We also update their blocking periods (in fact,

**Algorithm 1:** Blocking algorithm

---

**Scheduling routine**: At any time $t$, run an admitted and not yet completed job with shortest processing time.

**Event**: Upon release of a new job at time $t$:
Call **class admission routine**.

**Event**: Upon ending of a blocking or scheduling interval at time $t$:
Call **class admission routine**.

**Class admission routine**:
$i \leftarrow$ a shortest available job at $t$, i.e., $i \in \arg\min\{p_j \mid r_j \leq t \text{ and } d_j - t \geq (1+\delta)p_j\}$
$K \leftarrow$ the set of jobs whose scheduling intervals contain $t$
$j \leftarrow \arg\min\{p_k \mid k \in K\}$
**If** $i \in \mathcal{C}_c(j)$ and no $c' \geq c$ and $i' \in \mathcal{C}_{c'}(j)$ with $t \in B(i')$ exists, then
1. admit job $i$, $a_i = t$ and $e_i = a_i + (1+\delta)p_i$
   **if** $e_i \leq e_j$, then
   $f_i = \min\{e_j, e_i + \beta p_i\}$
   set $S(i) = [a_i, e_i)$ and $B(i) = [e_i, f_i)$
   **else**
   set $e_j = e_i$ and $f_i = e_i$
   **modify** $S(k)$ and $B(k)$ for $k \in K$
2. **update** $B(k)$ with $B(k) \cap [t, \infty) \neq \emptyset$ and $k \in \mathcal{C}_{c'}(j)$ for $c' < c$

---

single intervals) to reflect their new starting points. If the parent $\pi(k)$ of $k$ does not exist, $B(k)$ remains empty; otherwise we set $B(k) := [e_k, f_k)$ where $f_k = \min\{e_{\pi(k)}, e_k + \beta p_k\}$. Note that, after this update, the blocking intervals of any but the largest such job will be empty. Moreover, the just admitted job $i$ does not get a blocking period in this special case.

During the analysis of the blocking algorithm, we show that any admitted job $j$ still completes before $a_j + (1+\delta)p_j$ and that $e_j \leq a_j + (1+2\delta)p_j$ holds in retrospective for all admitted jobs $j$. Thus, any job $j$ that admits another job $i$ officially assigns this job a scheduling interval of length $(1+\delta)p_i$ but, for ensuring $j$'s completion, expects loosing $(1+2\delta)p_i$ time units of its scheduling interval $S(j)$. We summarize the blocking algorithm in Algorithm 1.

## 2.2 Results on the blocking algorithm

As noted in [8], it is sufficient to concentrate on instances with small slack. The same is true for our blocking algorithm as for $\varepsilon > 1$ we run the algorithm with $\varepsilon = 1$ and obtain constant competitive ratios. Thus, for the analysis we assume $0 < \varepsilon \leq 1$. Moreover, in the $\delta$-commitment model, committing to the completion of a job $j$ at an earlier point in time clearly satisfies committing at a remaining slack of $\delta p_j$. Therefore, we assume $\delta \in [\frac{\varepsilon}{2}, \varepsilon)$.

**Remark 1.** *The blocking algorithm guarantees to complete every job that it started. When the blocking algorithm commits to the completion of job $j$, this happens no later than $d_j - (1+\delta)p_j$. Hence, the algorithm is stricter than required by any of the two commitment models.*

For scheduling with commitment upon admission, we give an (up to constants) optimal online algorithm with competitive ratio $\Theta(1/\varepsilon)$. For scheduling with $\delta$-commitment, our result interpolates between the models commitment upon starting a job and commitment upon arrival. If $\delta \leq \varepsilon/2$, the competitive ratio collapses to $\Theta(1/\varepsilon)$ which is best possible [8].

For $\delta \to \varepsilon$, the commitment requirements essentially implies commitment upon job arrival which has unbounded competitive ratio [8].

**Theorem 1.** *Let $0 < \varepsilon \leq 1$. The blocking algorithm is $\mathcal{O}(\frac{\varepsilon}{(\varepsilon-\delta')\delta'})$-competitive for scheduling with commitment where $\delta' = \varepsilon/2$ in the commitment upon admission model and $\delta' = \max\{\delta, \varepsilon/2\}$ in the $\delta$-commitment model.*

The proof of Theorem 1 consists of two parts. In the first one, we show that the blocking algorithm completes all admitted jobs on time. The second part is to show that the blocking algorithm admits sufficiently many jobs to be competitive.

# 3 Completing all admitted jobs on time

We show that the blocking algorithm finishes every admitted job on time in Lemma 2. To this end, we need the following technical lemma about the length of the final scheduling interval of an admitted job $j$.

**Lemma 1.** *Let $0 < \delta < \varepsilon$ be fixed. If $\gamma > 0$ satisfies*

$$(1 + 2\delta)\gamma \leq \delta, \tag{1}$$

*then the length of the scheduling interval $S(j)$ of an admitted job $j$ is upper bounded by $(1 + 2\delta)p_j$. Moreover, the scheduling interval $S(j)$ contains the scheduling intervals of all its descendants.*

*Proof.* By definition of the blocking algorithm, the end point $e_j$ of the scheduling interval of job $j$ is only modified when $j$ or one of $j$'s descendants admits another job. Let us consider such a case: If job $j$ admits a job $i$ whose scheduling interval does not fit the scheduling interval of $j$, we set $e_j = e_i = a_i + (1 + \delta)p_i$ to accommodate the scheduling interval $S(i)$ within $S(j)$. The same modification is applied to any ancestor $k$ of $j$ with $e_k < e_i$. This implies that, after such a modification of the scheduling interval, neither $j$ nor any affected ancestors $k$ of $j$ are the smallest jobs in their scheduling intervals anymore. In particular, no job whose scheduling interval was modified in such a case at time $t$ is able to admit jobs after $t$. Hence, any job $j$ can only admit other jobs within the interval $[a_j, a_j + (1 + \delta)p_j)$. In particular, $a_i \leq a_j + (1 + \delta)p_j$ for any job $i$ with $\pi(i) = j$.

Thus, by induction, it is sufficient to show that $a_i + (1 + 2\delta)p_i \leq a_j + (1 + 2\delta)p_j$ for admitted jobs $i$ and $j$ with $\pi(i) = j$ in order to prove the lemma. Note that $\pi(i) = j$ implies $p_i < \gamma p_j$. Thus,

$$a_i + (1 + 2\delta)p_i \leq (a_j + (1 + \delta)p_j) + (1 + 2\delta)\gamma p_j \leq a_j + (1 + 2\delta)p_j,$$

where the last inequality follows from Equation (1). □

Scheduling in Shortest Processing Time order guarantees the following.

**Observation 1.** *If $j$ is the shortest job such that $t \in S(j)$, then $j$ has the highest scheduling priority among all admitted and not yet completed jobs.*

We proceed with proving that the blocking algorithm completes all admitted jobs before their deadlines.

**Theorem 2.** *Let $0 < \delta < \varepsilon$ be fixed. If $0 < \gamma < 1$ and $\beta \geq 1$ satisfy*

$$\frac{\beta/2}{\beta/2 + (1+2\delta)} \big(1 + \delta - 2(1+2\delta)\gamma\big) \geq 1, \tag{2}$$

*then the blocking algorithm will complete a job $j$ that was admitted at $a_j \leq d_j - (1+\delta)p_j$ on time.*

Our choice of parameters guarantees that Inequality (2) is satisfied.

*Proof.* Let $j$ be a job admitted by the blocking algorithm with $a_j \leq d_j - (1+\delta)p_j$. Hence, showing that a job $j$ completes before time $t_j := a_j + (1+\delta)p_j$ proves the theorem. By Observation 1, we know that each job $j$ has highest priority in its own scheduling interval if the time point does not belong to the scheduling interval of a descendant of $j$. Thus, it suffices to show that at most $\delta p_j$ units of time in $[a_j, t_j)$ belong to scheduling intervals $S(i)$ of descendants of $j$. By Lemma 1, the scheduling intervals of any descendant $k$ of a child $i$ of $j$ is contained in $S(i)$. Hence, it is sufficient to only consider $K$, the set of children of $j$. In order to bound the contribution of each child $i \in K$, we partition $K$ into two sets. The first set $K_1$ contains all children of $j$ that where admitted as the first jobs in their class $\mathcal{C}_c(j)$. The set $K_2$ contains the remaining jobs.

We start with $K_2$. Consider a job $i \in \mathcal{C}_c(j)$ admitted by $j$. By Lemma 1, we know that $|S(i)| = (1 + \mu\delta)p_i$ where $1 \leq \mu \leq 2$. Let $k \in \mathcal{C}_c(j)$ be the previous job admitted by $j$ in class $c$. Then, $B(k) \subseteq [e_k, e_i)$. Since scheduling and blocking periods of children of $j$ are always disjoint, $j$ had highest scheduling priority in $B(k)$. Hence, during $B(k) \cup S(i)$ job $j$ was processed for at least $|B(k)|$ units of time. In other words, $j$ used a fraction of $\frac{|B(k)|}{|B(k) \cup S(i)|}$ of $B(k) \cup S(i)$. We can rewrite this ratio by

$$\frac{|B(k)|}{|B(k) \cup S(i)|} = \frac{\beta p_k}{\beta p_k + (1+\mu\delta)p_i} = \frac{\nu\beta}{\nu\beta + (1+\mu\delta)},$$

where $\nu := \frac{p_k}{p_i} \in (\frac{1}{2}, 2]$. By differentiating with respect to $\nu$ and $\mu$, we observe that the last term is increasing in $\nu$ and decreasing in $\mu$. Thus, we can lower bound this expression by

$$\frac{|B(k)|}{|B(k) \cup S(i)|} \geq \frac{\frac{\beta}{2}}{\frac{\beta}{2} + (1+2\delta)}. \tag{3}$$

Therefore, in $\bigcup_{i \in K} B(i) \cup \bigcup_{i \in K_2} S(i)$, $j$ uses at least a $\frac{\beta/2}{\beta/2 + (1+2\delta)}$-fraction for processing.

We proceed with considering the set $K_1$. The total processing volume of first jobs is bounded by

$$\sum_{c=0}^{\infty} \frac{\gamma}{2^c} p_j = 2\gamma p_j.$$

By Lemma 1, we know that $|S(i)| \leq (1+2\delta)p_i$. Combining these two observations, we can upper bound the contribution of $K_1$ by

$$\Big| \bigcup_{i \in K_1} S(i) \Big| \leq 2(1+2\delta)\gamma p_j. \tag{4}$$

Combining Equations (3) and (4), we conclude that $j$ is scheduled for at least

$$\left|[a_j, t_j) \setminus \bigcup_{i \in K} S(i)\right| \geq \frac{\beta/2}{\beta/2 + (1+2\delta)} \left((1+\delta) - 2(1+2\delta)\gamma\right) p_j \geq p_j$$

units of time, where the last inequality follows from Equation (2). Thus, $j$ completes before $t_j = a_j + (1+\delta)p_j \leq d_j$. $\square$

# 4 Admitting sufficiently many jobs

After proving that each admitted job completes on time, we prove in this section that the blocking algorithm admits enough jobs to be $\mathcal{O}(\frac{\varepsilon}{(\varepsilon-\delta)\delta})$-competitive.

## 4.1 Key lemma on the size of non-admitted jobs

For the proof of the main result in this section, we rely on the following strong, structural lemma. It relates the volume that any feasible schedule $\sigma$ processes in some interval to the size of jobs admitted by an online algorithm ALG satisfying the following two properties: (i) ALG never admits a job $j$ later than $d_j - (1+\delta)p_j$ for $0 < \delta < \varepsilon$ and (ii) at time $t$, the algorithm admits any job $j$ with $p_j \leq u_t$ for $u_t \in (0, \infty]$. Note that our blocking algorithm as well as the region algorithm in [8] satisfy (i) and (ii). Let $X^\sigma$ be the set of jobs that $\sigma$ completed and that ALG did not admit. Let $C_x$ be the completion time of job $x \in X^\sigma$ in $\sigma$.

**Lemma 2.** *Let $0 \leq t_1 \leq t_2$ and fix $x \in X^\sigma$ as well as $Y \subset X^\sigma \setminus \{x\}$. If*

*(R) $r_x \geq t_1$ as well as $r_y \geq t_1$ for all $y \in Y$,*

*(C) $C_x \geq C_y$ for all $y \in Y$, and*

*(P) $\sum_{y \in Y} p_y \geq \frac{\varepsilon}{\varepsilon-\delta}(t_2 - t_1)$*

*hold, then $p_x \geq u_{t_2}$ where $u_{t_2}$ is the upper bound imposed by ALG at time $t_2$.*

*Proof.* We show the lemma by contradiction. More precisely, we show that, if $p_x < u_t$, the schedule $\sigma$ cannot complete $x$ on time and, hence, is not feasible.

Remember that $x \in X^\sigma$ implies that the blocking algorithm did not admit job $x$ at any point $t$. At time $t_2$, there are two possible reasons why $x$ was not admitted: $p_x \geq u_t$ or $d_x - t_2 < (1+\delta)p_x$. In case of the former, the statement of the lemma holds. Thus, let us assume $p_x < u_t$ and, therefore, $d_x - t_2 < (1+\delta)p_x$ holds. As job $x$ arrived with a slack of at least $\varepsilon p_x$ at its release date $r_x$ and $r_x \geq t_1$ by assumption, we have

$$t_2 - t_1 \geq t_2 - d_x + d_x - r_x > -(1+\delta)p_x + (1+\varepsilon)p_x = (\varepsilon - \delta)p_x.$$

As all jobs in $Y$ complete earlier than $x$ by Assumption (C) and are only released after $t_1$ by (R), the volume processed by $\sigma$ in $[t_1, C_x)$ is greater than $\frac{\varepsilon}{\varepsilon-\delta}(t_2 - t_1) + p_x$ by (P). Moreover, $\sigma$ can process at most $t_2 - t_1$ in $[t_1, t_2)$. This implies that $\sigma$ has to process job parts with a processing volume of at least

$$\frac{\delta}{\varepsilon-\delta}(t_2 - t_1) + p_x > \frac{\delta}{\varepsilon-\delta}(\varepsilon-\delta)p_x + p_x = (1+\delta)p_x$$

after time $t_2$. Thus, $C_x > t_2 + (1+\delta)p_x > d_x$ which contradicts the feasibility of $\sigma$.

$\square$

Observe that the online algorithm ALG admits any job that satisfies $p_j \le u_t$. In particular, if $u_t = \infty$ for some time point $t$, ALG admits *any* job. This implies the following corollary.

**Corollary 1.** *If $u_{t_2} = \infty$, there does not exist a job $x \in X^\sigma$ such that there exists a time $0 \le t_1 \le t_2$ and a set $Y \subset X^\sigma \setminus \{x\}$ satisfying (R), (C), and (P) in Lemma 2.*

## 4.2 The blocking algorithm is $\mathcal{O}(\frac{\varepsilon}{(\varepsilon-\delta)\delta})$-competitive

**Theorem 3.** *An optimal (offline) algorithm can complete at most a factor $\alpha + 4$ more jobs on time than admitted by the blocking algorithm where $\alpha := \frac{\varepsilon}{\varepsilon-\delta}\big(2\beta + \frac{1+2\delta}{\gamma}\big)$.*

For proving the theorem, we fix an instance and an optimal offline algorithm OPT. Let $X$ be the jobs that OPT scheduled and the blocking algorithm did not admit. We assume without loss of generality that OPT completes all jobs in $X$ on time. Further, let $J$ be the jobs that the blocking algorithm scheduled. Then, $X \cup J$ clearly is a superset of the jobs that OPT scheduled. Hence, to show the theorem it is sufficient to prove that $|X| \le (\alpha + 3)|J|$.

More precisely, we develop a charging scheme of jobs in $X$ to the intervals that the algorithm created such that no interval gets charged too many jobs. We then bound the number of intervals in terms of $|J|$. The idea behind our charging scheme is that OPT is not able to schedule arbitrary many jobs during a scheduling interval or a blocking period created by the blocking algorithm.

Intuitively, jobs that were released during a scheduling interval or a blocking period and not admitted by the algorithm have to satisfy certain lower bound on their processing times. Thus, the charging scheme relies on the release date $r_x$ and the size $p_x$ of a job $x \in X$ as well as on the precise structure of the intervals created by the blocking algorithm. The number of jobs we charge to one interval will depend on the relative length of the interval.

We retrospectively consider $\mathcal{I}$, the set of all the intervals the blocking algorithm created, i.e., the scheduling intervals $S(j)$ and the (possibly many) intervals that form the blocking periods $B(j)$. For simplicity we say $i \in \mathcal{I}$ for an interval $I_i = [\mu_i, \nu_i)$. We say job $j$ *owns* or is the *owner* of interval $I_i$ if either $I_i$ is the scheduling interval of $j$ or $I_i$ is one of the intervals that form the blocking period of $j$. We also define $j(i)$ to be the job that owns a particular interval $I_i$. If $I_i$ is the scheduling interval of $j(i)$, we say that $I_i$ is of *type S*. Otherwise, the interval belongs to the blocking period $B(j)$ and is called of *type B*. We collect the indices of the intervals that form the blocking period $B(j)$ of job $j$ in the set $\mathcal{B}(j)$. In Lemma 4, we show that a scheduling interval $S(j)$ gets at most $\frac{\varepsilon}{\varepsilon-\delta}\frac{1+2\delta}{\gamma} + 1$ many jobs and a blocking period $B(j)$ is assigned at most $2\frac{\varepsilon}{\varepsilon-\delta}\beta + |\mathcal{B}(j)|$ jobs. Lemma 3 shows that the total number of intervals that the blocking algorithm created is bounded from above by $3|J|$. Combining these two lemmas then proves that indeed $|X| \le (\alpha + 3)|J|$.

**Lemma 3.** *The blocking algorithm creates at most $3|J|$ intervals.*

*Proof.* The blocking algorithm does not split scheduling intervals. Hence, the number of scheduling intervals is exactly $|J|$. The admission of a new job splits at most one interval of a blocking period into two intervals. Thus, the admission of $|J|$ jobs creates at most $2|J| - 1$ intervals of type $B$. In total, the algorithm creates at most $3|J| - 1$ intervals. □

The charging scheme developed in Lemma 4 is based on a careful modification of the following partition $(F_i)_{i \in \mathcal{I}}$. Fix an interval $I_i$, its owner $j$, and the parent $\pi$ of $j$. We can now define the set $F_i \subset X$ that contains all jobs $x \in X$ that are released during $I_i$ and satisfy certain bounds on their processing times. If $I_i$ is of type $S$, the owner $j$ immediately admits any job $x$ with $r_x \in I_i$ that is smaller than $\gamma p_j$ unless the time also belongs to the scheduling interval or the blocking period of a descendant of $j$. Hence, we use $\gamma p_j$ as a natural lower bound for jobs in $F_i$. Similarly, $\gamma p_\pi$ is a natural upper bound as such jobs can also be assigned to $F_{i'}$ where $I_{i'} = S(\pi)$. If $I_i$ is of type $B$, a similar argumentation shows that $p_x \geq p_j/2$ and $p_x < \gamma p_\pi$ are natural bounds on the processing time. Hence,

$$F_i := \begin{cases} \{x \in X : r_x \in I_i \text{ and } \gamma p_j \leq p_x < \gamma p_\pi\} & \text{if } i \text{ is of type } S, \\ \{x \in X : r_x \in I_i \text{ and } p_j/2 \leq p_x < \gamma p_\pi\} & \text{if } i \text{ is of type } B. \end{cases}$$

As argued above, jobs $x \in X$ with $r_x \in S(j)$ and $p_x < \gamma p_j$ are released within a blocking or a scheduling interval of one of the descendants of $j$ as otherwise the blocking algorithm admits such jobs. Thus, such jobs $x$ belong to the set $F_{i'}$ of the corresponding interval $I_{i'}$. Moreover, jobs with release dates not within any blocking or scheduling interval of some job $j$ are admitted by the blocking algorithm. Hence, the following observation holds.

**Observation 2.** $X = \bigcup_{i \in \mathcal{I}} F_i$.

We now formalize how many jobs in $X$ we will assign to a specific interval $I_i = [\mu_i, \nu_i)$ with owner $j$. Depending on the type of $I_i$ we define

$$\lambda_i = \begin{cases} \gamma p_j & \text{if } i \text{ is of type } S \\ \frac{p_j}{2} & \text{if } i \text{ is of type } B. \end{cases}$$

Then, the *target number* of interval $I_i$ is defined as $\varphi_i := \left\lfloor \frac{\varepsilon}{\varepsilon-\delta} \frac{\nu_i - \mu_i}{\lambda_i} \right\rfloor + 1$. For a scheduling interval $S(j) = I_i$, this implies that the target number is

$$\varphi_i = \left\lfloor \frac{\varepsilon}{\varepsilon - \delta} \frac{\nu_i - \mu_i}{\gamma p_j} \right\rfloor + 1 \leq \frac{\varepsilon}{\varepsilon - \delta} \frac{1 + 2\delta}{\gamma} + 1$$

by Lemma 1. For a blocking period $B(j) = \bigcup_{i \in \mathcal{B}(j)} I_{i_k}$ the target number is

$$\sum_{i \in \mathcal{B}(j)} \varphi_i = \sum_{i \in \mathcal{B}(j)} \left( \left\lfloor \frac{\varepsilon}{\varepsilon - \delta} \frac{\nu_i - \mu_i}{p_j/2} \right\rfloor + 1 \right) \leq |\mathcal{B}(j)| + \sum_{i \in \mathcal{B}(j)} \frac{\varepsilon}{\varepsilon - \delta} \frac{\nu_i - \mu_i}{p_j/2} \leq |\mathcal{B}(j)| + 2\frac{\varepsilon}{\varepsilon - \delta}\beta$$

because $\beta p_j \geq |B(j)| = \sum_{i \in \mathcal{B}(j)} (\nu_i - \mu_i)$.

If each of the sets $F_i$ satisfies $|F_i| \leq \varphi_i$, Observation 2 and Lemma 3 guarantee $|X| \leq \left(\frac{\varepsilon}{\varepsilon-\delta}(2\beta + \frac{1+2\delta}{\gamma}) + 3\right)|J|$ and, hence, prove Theorem 1. In general, this does not have to be true as Opt may preempt jobs and process the parts during several intervals $I_i$. In the remainder of this section, we show that there exists another partition $\left(G_i\right)_{i \in \mathcal{I}}$ of the jobs in $X$ such that $|G_i| \leq \varphi_i$ holds.

**Lemma 4.** $|X| \leq \alpha|J| + |\mathcal{I}|$

*Proof.* As observed before it suffices to show that there is a partition $\mathcal{G} = \left(G_i\right)_{i \in \mathcal{I}}$ such that $|G_i| \leq \varphi_i$ and $\bigcup_{i \in \mathcal{I}} G_i = X$ in order to prove the lemma. The high level idea of this proof is the following: Consider an Interval $I_i = [\mu_i, \nu_i)$. If $F_i$ does not contain too many jobs, i.e., $|F_i| \leq \varphi_i$, we would like to set $G_i = F_i$. Otherwise, we find another interval $I_{i'}$ with $|F_{i'}| < \varphi_{i'}$

that has a later end point than $I_i$, i.e., $\nu_i \leq \nu_{i'}$, such that we can assign the excess jobs in $F_i$ to $I_{i'}$.

In order to repeatedly apply Lemma 2, we only assign such excess jobs $x \in F_i$ to $G_{i'}$ if their processing time is at least the lower bound on the size of jobs in $F_{i'}$, i.e., $p_x \geq \lambda_{i'}$. Then, by our choice of parameters, a set $G_{i'}$ with $\varphi_{i'}$ many jobs of size at least $\lambda_{i'}$ "covers" the interval $I_{i'} = [\mu_{i'}, \nu_{i'})$ as often as required by (iii) in Lemma 2, i.e.,

$$\sum_{x \in G_{i'}} p_x \geq \varphi_{i'} \cdot \lambda_{i'} = \frac{\varepsilon}{\varepsilon - \delta}(\nu_{i'} - \mu_{i'}). \tag{5}$$

The proof consists of two parts: the first one is to inductively construct the partition $\mathcal{G} = \big(G_i\big)_{i \in \mathcal{I}}$ of $X$ with $|G_i| \leq \varphi_i$. The second one is the proof that a job $x \in G_i$ satisfies $p_x \geq \lambda_i$. During the construction of $\mathcal{G}$ we define temporary sets $A_i \subset X$ for intervals $i \in \mathcal{I}$. The set $G_i$ will be chosen as a subset of $F_i \cup A_i$ of appropriate size. In order to apply Lemma 2 to each job in $A_i$ individually, alongside $A_i$, we will construct a set $Y_{x,i}$ and a time $t_{x,i} \leq r_x$ for each job $x \in X$ that is added to $A_i$. Let $C_x^*$ be the completion time of some job $x \in X$ in the optimal schedule Opt. In the second part of the proof we will then show that $x$, $t_{x,i}$, and $Y_{x,i}$ satisfy

(R) $r_y \geq t_{x,i}$ for all $y \in Y_{x,i}$,

(C) $C_x^* \geq C_y^*$ for all $y \in Y_{x,i}$, and

(P) $\sum_{y \in Y(x,i)} p_y \geq \frac{\varepsilon}{\varepsilon - \delta}(\mu_i - t_{x,i})$.

Then, $x$, $Y = Y_{x,i}$, $t_1 = t_{x,i}$, and $t_2 = \mu_i$ satisfy the conditions of Lemma 2 and we can deduce that the processing time of $x$ is at least as large as the lower bound of interval $i$, i.e., $p_x \geq \lambda_i$.

**Constructing $\boldsymbol{\mathcal{G} = \big(G_i\big)_{i \in \mathcal{I}_M}}$.** We follow the order $\leq_\nu$ that is defined by $\nu_i$, i.e., the end points of the intervals. For two intervals $I_{i'}$ and $I_i$ with $\nu_{i'} = \nu_i$, there has to be a difference in the size of their owners $j(i')$ and $j(i)$. We use this to break ties, i.e., if $p_{j(i)} < p_{j(i')}$, then $I_i <_\nu I_{i'}$. Then, $\leq_\nu$ is a total order on the set of intervals $\mathcal{I}$. We index the intervals with respect to the total order $\leq_\nu$ and abbreviate $I_i \leq_\nu I_{i'}$ by $i \leq i'$. For simplicity, we include a machine job $M$ with infinite processing time, i.e., $I_M = (-\infty, \infty)$. By Observation 2, we know that $F_M = \emptyset$. Additionally, we set $\lambda_M = \varphi_M = \infty$. Clearly, $M$ is the maximal element in the just defined order.

We start by setting $A_i = \emptyset$ for all intervals $i \in \mathcal{I}_M$. For simplicity, we define $Y_{x,i} = \emptyset$ for each job $x \in X$ and each interval $i$. The preliminary value of the time $t_{x,i}$ is the minimum of the start point $\mu_i$ of the interval $i$ and the release date $r_x$ of $x$, i.e., $t_{x,i} := \min\{\mu_i, r_x\}$. The order of the construction follows the total order defined above. We refer by *step $i$* to the step in the construction where $G_i$ was defined.

Let $I_i$ be the next interval to consider during the construction. Let $i' \geq i$ be the first interval that contains the end point of $I_i$, i.e., $\nu_i \in I_{i'}$. This interval $I_{i'}$ does exist as the last interval in our order is the machine interval $S(M) = (-\infty, \infty)$. Depending on the cardinality of $F_i \cup A_i$, we have to distinguish two cases. If $|F_i \cup A_i| \leq \varphi_i$, we set $G_i = F_i \cup A_i$.

If $|F_i \cup A_i| > \varphi_i$, we order the jobs in $F_i \cup A_i$ in increasing completion times in Opt. The first $\varphi_i$ jobs are assigned to $G_i$ while the remaining $|F_i \cup A_i| - \varphi_i$ jobs are added to $A_{i'}$. In this case, we might have to redefine the times $t_{x,i'}$ and the sets $Y_{x,i'}$ for the jobs $x$ that were newly added to $A_{i'}$. Fix such a job $x$. If there is no job $z$ in the just defined set $G_i$ that has a smaller release date than $t_{x,i}$, we set $t_{x,i'} = t_{x,i}$ and $Y_{x,i'} = Y_{x,i} \cup G_i$. Otherwise let $z \in G_i$ be a job with $r_z \leq t_{x,i}$ that has the smallest time $t_{z,i}$. We set $t_{x,i'} = t_{z,i}$ and $Y_{x,i'} = Y_{z,i} \cup G_i$.

**Bounding the size of the jobs in $G_i$.** We consider the intervals again in the order defined by the endpoints and show by induction on the interval indices that any job $x$ in $G_i$ indeed satisfies $p_x \geq \lambda_i$. Clearly, if $x \in F_i \cap G_i$, the size bound is fulfilled by definition of the set $F_i$. Hence, in order to show the lower bound on the processing time of $x \in G_i$, it is sufficient to consider jobs in $G_i \setminus F_i \subset A_i$. To this end, we show that (R), (C), and (P) are satisfied. Then, Lemma 2 guarantees that $p_x \geq \lambda_i$.

Let $i$ be the smallest index such that $A_i \neq \emptyset$ and fix $x \in A_i$. Let $i' < i$ be such that $x \in F_{i'}$. Then, $\nu_{i'} \in I_i$ by construction and we assigned $x$ in step $i'$ to $A_i$. Due the choice of $i$, we know that $G_{i'} \subset F_{i'}$. This implies that (R) is trivially satisfied for the time $t_{x,i'} = \min\{\mu_{i'}, r_x\}$ and that $p_y \geq \lambda_{i'}$ for each $y \in G_{i'}$. For (P), consider that $|G_{i'}| = \varphi_{i'}$. It follows

$$\sum_{y \in G_{i'}} p_y \geq \varphi_{i'} \cdot \lambda_{i'} \geq \frac{\varepsilon}{\varepsilon - \delta}(\nu_{i'} - \mu_{i'}) = \frac{\varepsilon}{\varepsilon - \delta}(\nu_{i'} - t_{x,i'}) \geq \frac{\varepsilon}{\varepsilon - \delta}(\mu_i - t_{x,i})$$

as $\nu_{i'} \in I_i$ implies $\nu_{i'} \geq \mu_i$. By construction, (C) is also true because $x$ completes later than any of the first $\varphi_{i'}$ jobs in $F_{i'}$ that were assigned to $G_{i'}$. As discussed before, this implies that indeed $p_x \geq \lambda_i$.

Assume that the conditions (R), (C), and (P) are satisfied for all $x \in A_i$ for all $1 \leq i < h$. This implies that, for $i < h$, the set $G_i$ only contains jobs with $p_x \geq \lambda_i$. Let $i \geq h$ be the first index with $A_i \neq \emptyset$ and fix $x \in A_i$. We want to show that $p_x \geq \lambda_i$. Let $i' < i$ be maximal such that $x \in A_{i'} \cup F_{i'}$. By induction, $p_y \geq \lambda_{i'}$ holds for all $y \in G_{i'}$. Because $x$ did not fit in $G_{i'}$ anymore, $|G_{i'}| = \varphi_{i'}$.

As before, we consider two different cases depending on the jobs in $G_{i'}$. If there is no $z \in G_{i'}$ with $r_z < t_{x,i'} = t_{x,i}$, (R) and (C) are trivially satisfied by construction and by induction. For (P), consider

$$\begin{aligned}
\sum_{y \in Y_{x,i}} p_y &= \sum_{y \in Y_{x,i'}} p_y + \sum_{y in G_{i'}} p_y \\
&\geq \frac{\varepsilon}{\varepsilon - \delta}(\mu_{i'} - t_{x,i'}) + \lambda_{i'} \cdot \varphi_{i'} \\
&\geq \frac{\varepsilon}{\varepsilon - \delta}(\mu_{i'} - t_{x,i}) + \frac{\varepsilon}{\varepsilon - \delta}(\nu_{i'} - \mu_{i'}) \\
&\geq \frac{\varepsilon}{\varepsilon - \delta}(\mu_i - t_{x,i}).
\end{aligned}$$

Here, the first inequality follows from induction on $x$, $Y_{x,i'}$, and $t_{x,i'}$, the second by definition, and the third inequality follows from $\nu_{i'} \geq \mu_i$.

If there is a job $z \in G_{i'}$ with $r_z < t_{x,i'} \leq \mu_{i'}$, then $z \in A_{i'}$. During the construction of $G_i$, we chose $z$ with minimal $t_{z,i'}$. We have that $r_y \geq t_{y,i'} \geq t_{z,i'}$ for all $y \in G_{i'}$ and $r_x \geq t_{x,i'} > r_z \geq t_{z,i'}$. Moreover, by induction, $r_y \geq t_{y,i'}$ holds for all $y \in Y_{z,i'}$. Thus, $t_{x,i}$ and $Y_{x,i}$ satisfy (R). For (C), consider that $C_x^* \geq C_y^*$ for all $y \in G_{i'}$ by construction and, thus,

$C_x^* \geq C_z^* \geq C_y^*$ also holds for all $y \in Y_{z,i'}$. For (P), observe that

$$\begin{aligned}
\sum_{y \in Y_{x,i}} p_y &= \sum_{y \in Y_{z,i'}} p_y + \sum_{y \in G_{i'}} p_y \\
&\geq \frac{\varepsilon}{\varepsilon - \delta}(\mu_{i'} - t_{z,i'}) + \lambda_{i'} \cdot \varphi_{i'} \\
&\geq \frac{\varepsilon}{\varepsilon - \delta}(\mu_{i'} - t_{x,i}) + \frac{\varepsilon}{\varepsilon - \delta}(\nu_{i'} - \mu_{i'}) \\
&\geq \frac{\varepsilon}{\varepsilon - \delta}(\mu_i - t_{x,i}).
\end{aligned}$$

Here, the first inequality follows by induction on $z$, $Y_{z,i'}$, and $t_{z,i'}$, the second by definition of $\lambda_{i'}$ and $\varphi_{i'}$, and the last inequality is due to $\nu_{i'} \in I_i$ which in turn implies $\nu_{i'} \geq \mu_i$.

**Showing $|X| \leq \alpha|J| + |\mathcal{I}|$.** By construction, we know that $\bigcup_{i \in \mathcal{I}_M} G_i = X$. Thus, the claim follows if $G_M = \emptyset$, i.e., if the machine interval $M$ did not receive any job $x \in X$. The machine interval $M$ is the last interval we consider during the construction. As the algorithm admits any job that is not released within the scheduling interval or the blocking period of another job, we know that $F_M = \emptyset$ by definition. Hence, $G_M \neq \emptyset$ implies that $A_M \neq \emptyset$. As (R), (C), and (P) also hold for any job $x \in A_M$, Corollary 1 implies that such an $x$ cannot exist. Hence, the number of jobs in $X$ is indeed bounded by $\alpha|J| + |\mathcal{I}|$.

□

**Proving Theorem 3**

*Proof of Theorem 3.* As discussed before, the union $X \cup J$ of $X$, the jobs only scheduled by Opt, and $J$, the jobs admitted by the blocking algorithm, is a superset of the jobs that Opt completed. Lemma 4 shows that $|X| \leq \frac{\varepsilon}{\varepsilon-\delta}\left(2\beta + \frac{1+2\delta}{\gamma}\right)|J| + |\mathcal{I}|$. Combining this with the bound on $\mathcal{I}$ given in Lemma 3, we conclude

$$\textsc{Opt} \leq \left(\frac{\varepsilon}{\varepsilon - \delta}\left(2\beta + \frac{1 + 2\delta}{\gamma}\right) + 3\right)|J|.$$

□

**Completing the proof of Theorem 1**

*Proof of Theorem 1.* In Theorem 2 we show that the blocking algorithm completes all admitted jobs $J$ on time. First, this implies that the blocking algorithm is feasible for the model commitment upon admission. Second, as no job $j \in J$ is admitted later than $d_j - (1 + \delta)p_j$ this shows that the blocking algorithm also solves scheduling with $\delta$-commitment. Third, in combination with Theorem 3, where we bound an optimal solution Opt in terms of $|J|$, this shows that the blocking algorithm achieves a competitive ratio of

$$\alpha = \left(\frac{\varepsilon}{\varepsilon - \delta}\left(2\beta + \frac{1 + 2\delta}{\gamma}\right) + 3\right).$$

By our choice of parameters $\beta = \frac{16}{\delta}$ and $\gamma = \frac{\delta}{16}$ we have that $\alpha \in \mathcal{O}\left(\frac{\varepsilon}{(\varepsilon-\delta)\delta}\right)$ which completes the proof of Theorem 1. □

## 5 Easier analysis of the region algorithm

In [8], we give an algorithmic framework that handles various commitment models in online throughput maximization. First, it is best possible when maximizing throughput without any commitment requirements. Second, it gives the first non-trivial competitive ratio when scheduling with commitment upon starting a job or with $\delta$-commitment.

The region algorithm uses two parameters, $\alpha \geq 1$ and $\beta \ll 1$ to implement the different commitment models. When admitting a job $j$, the algorithm assigns it a region $R(j)$ of length $\alpha p_j$ during which only jobs of size at most $\beta p_j$ are admitted. The region algorithm never admits a job later than $d_j - (1+\delta)p_j$. As regions are interrupted for the admission of smaller jobs and resumed after the end of their regions, in the end, the region algorithm creates disjoints intervals that belong to the region $R(j)$. With Lemma 2, which is a strong generalization of the Volume Lemma (Lemma 3) in [8], we can now simplify the proof of Theorem 5.

**Theorem 4** (Theorem 5 in [8])**.** *The number of jobs that an optimal (offline) algorithm can complete on time is by at most a multiplicative factor* $\lambda + 2$ *larger than the number of jobs admitted by the region algorithm, where* $\lambda := \frac{\varepsilon}{\varepsilon-\delta}\frac{\alpha}{\beta}$*, for* $0 < \delta < \varepsilon \leq 1$*.*

As before, fix an optimal (offline) solution OPT and let $X$ be the set of jobs only processed by OPT but not by the region algorithm. Without loss of generality, we assume that OPT completes every job that it starts. Let $J$ be the set of jobs admitted by the region algorithm. Then, $X \cup J$ is a superset of the jobs completed by OPT and showing $|X| \leq |J|+2$ is sufficient to prove the theorem. We develop a new charging scheme of the jobs in $X$ *directly* to the intervals created by the region algorithm in Lemma 6 and bound the number of intervals in Lemma 5.

Let $\mathcal{I}$ be the set of disjoint intervals created by the region algorithm. Then, the number of intervals created by the region algorithm is bounded in terms of the number of jobs admitted by the region algorithm.

**Lemma 5.** *The region algorithm creates at most* $2|J|$ *intervals.*

*Proof.* The region algorithm splits at most one interval of a region into two intervals when admitting a new job. Thus, the admission of $|J|$ jobs creates at most $2|J|-1$ intervals. □

In order to give the simpler proof for Theorem 4, we start with a partition of the jobs in $X$ that is based on their release dates. If this partition does not satisfy the size bounds, we carefully modify the partition and repeatedly apply Lemma 2.

Fix an interval $I_i = [\mu_i, \nu_i)$ and its owner $j$. Then, analogously to Section 4, we define $F_i \subset X$ to contain all jobs $x$ that were released during $I_i$. The region algorithm admits any job released in $I_i$ that is smaller than $\beta p_{j(i)}$. Hence, defining $\lambda_i = \beta p_{j(i)}$ provides a natural lower bound on the jobs in $F_i$. In combination with the fact that the region algorithm never admits jobs too close to their deadlines, i.e., only before $d_j - (1+\delta)p_j$, we can directly apply Lemma 2 given that the conditions are met. Note that the intervals created by the region algorithm are always disjoint.

For $I_i = [\mu_i, \nu_i)$, we define $\varphi_i = \left\lfloor \frac{\nu_i - \mu_i}{\lambda_i} \right\rfloor + 1$ as the *target number*, i.e., the number of jobs we assign to interval $I_i$. Then, the number of jobs assigned to a region $R(j)$ is bounded by $\frac{\alpha}{\beta} + \rho_j$ where $\rho_j$ is the number of intervals belonging to $R(j)$. In combination with Lemma 5 that bounds the number of intervals, this gives us an upper bound on $|X|$, the number of jobs only completed by OPT.

**Lemma 6.** $|X| \leq \frac{\alpha}{\beta}|J| + |\mathcal{I}|$.

*Proof.* The proof follows the line of proof for Lemma 4. We construct a partition $\mathcal{G} = (G_i)_{i \in \mathcal{I}}$ that satisfies $|G_i| \leq \varphi_i$ for all $i \in \mathcal{I}$. As discussed above, this is sufficient to show the statement.

The structure of intervals created by the region algorithm is much simpler than the structure created by the blocking algorithm as all intervals are disjoint. This allows us to individually consider each maximal subset $\mathcal{I}'$ of $\mathcal{I}$ such that the intervals in $\mathcal{I}'$ form one contiguous interval in the analysis of the region algorithm. From now on, we assume that the intervals in $\mathcal{I}$ indeed form one interval.

The starting point of the construction of $\mathcal{G}$ is the partition $(F_i)_{i \in \mathcal{I}}$. If the set $F_i$ contains no more than $\varphi_i$ elements, we would like to set $G_i = F_i$. If the set $F_i$ contains too many elements, we move the excess jobs to some later interval $I_{i'}$, i.e., to an interval with $\nu_{i'} > \nu_i$. More precisely, we add such jobs to the set $A_{i'}$, that satisfies $p_x \geq \lambda_{i'}$ for all jobs $x \in A_{i'}$, and consider the set $A_{i'}$ when we define $G_{i'}$.

In order to show the size bound on jobs in some set $A_i$, we define a time $t_{x,i}$ and a set $Y_{x,i}$ for each job $x$ that is added to the set $A_i$. We guarantee that $t_{x,i}$ and $Y_{x,i}$ satisfy

(R) $r_y \geq t_{x,i}$ for all $y \in Y_{x,i}$,

(C) $C^*_x \geq C^*_y$ for all $y \in Y_{x,i}$, and

(P) $\sum_{y \in Y(x,i)} p_y \geq \frac{\varepsilon}{\varepsilon - \delta}(\mu_i - t_{x,i})$.

This allows us to apply Lemma 2 and deduce that indeed $p_x \geq \lambda_i$ holds.

The proof consists of two separate steps; the first one is the construction of $\mathcal{G}$ and the second one is showing $p_x \geq \lambda_i$ for $x \in A_i$. During these two steps we follow the order on the intervals defined by their end points. Note that, as the intervals created by the region algorithm are disjoint, this already defines a total order. From now on, we index the intervals in this order. The simpler structure of intervals created by the region algorithm implies that $\nu_i = \mu_{i+1}$ for $1 \leq i < |I|$. Moreover, we add a machine interval $I_M = [\nu_{|\mathcal{I}|}, \infty)$, which is the last element in the just defined order. As the region algorithm admits all jobs released not within the region of any already admitted job, we know that $F_M = \emptyset$. For simplicity, we additionally set $\lambda_M = \infty$.

**Constructing $\mathcal{G} = (G_i)_{i \in \mathcal{I}}$.** We start by defining $A_i = \emptyset$ for all intervals. Moreover, we temporarily set $Y_{x,i} = \emptyset$ and $t_{x,i} = \min\{\mu_i, r_x\}$ for each interval $i$ and each job $x$.

Let $i \in \mathcal{I}$ be the next interval to be considered during the construction. If $|F_i \cup A_i| \leq \varphi_i$, we set $G_i = F_i \cup A_i$. Otherwise, we sort the jobs in $x$ in order of their completion times $C^*_x$ in OPT. The set $G_i$ consists of the first $\varphi_i$ jobs and the remaining $|F_i \cup A_i| - \varphi_i$ jobs are added to the set $A_{i+1}$. We may have to redefine $Y_{x,i+1}$ and $t_{x,i+1}$ in this case. Fix a newly added job $x$. If no job $z \in G_i$ is released before $t_{x,i}$ we define $t_{x,i+1} = t_{x,i}$ and $Y_{x,i+1} = Y_{x,i} \cup G_i$. Otherwise let $z \in G_i$ with $r_z \leq t_{x,i}$ be the job with the smallest time $t_{z,i}$ and set $t_{x,i+1} = t_{z,i}$ and $Y_{x,i+1} = Y_{z,i} \cup G_i$.

**Bounding the size of jobs in $G_i$.** More precisely, we inductively show that each job in $A_i$ satisfies $p_x \geq \lambda_i$. During the induction, we follow again the order on $\mathcal{I}$ defined by the endpoints of the intervals. Let $i$ be the smallest index such that $A_i \neq \emptyset$ and fix $x \in A_i$. As $\nu_i = \mu_{i-1}$, we have that $G_{i-1} \subset F_{i-1}$ and $x \in F_{i-1}$ by choice of $i$. Therefore, $t_{x,i-1} = \mu_{i-1} \leq r_y$ holds for all $y \in G_{i-1} \subset F_{i-1}$ which implies (R). Moreover, $p_y \geq \lambda_{i-1}$ for $y \in G_{i-1}$ and $|G_{i-1}| = \varphi_{i-1}$. Hence,

$$\sum_{y \in G_{i-1}} p_y \geq \varphi_{i-1} \cdot \lambda_{i-1} \geq \frac{\varepsilon}{\varepsilon - \delta}(\nu_{i-1} - \mu_{i-1}) = \frac{\varepsilon}{\varepsilon - \delta}(\mu_i - t_{x,i-1}),$$

which implies (P). Condition (C) holds by construction. Thus, Lemma 2 implies that $p_x \geq \lambda_i$.

Let conditions (R), (P), and (C) be satisfied for all $x \in A_i$ for all $1 \leq i \leq h$. Note that this implies that the set $G_i$ only consists of jobs that satisfy $p_x \geq \lambda_i$ for all $1 \leq i \leq h$. Let $i > h$ be the next index with $A_i \neq \emptyset$ and fix $x \in A_i$. Then, $x \in F_{i-1} \cup A_{i-1}$. By construction we have that $|G_{i-1}| = \varphi_{i-1}$.

We distinguish two different cases depending on the jobs in $G_{i-1}$. If there is no $z \in G_{i-1}$ with $r_z < t_{x,i-1} = t_{x,i}$, the conditions (R) and (C) are trivially satisfied. For (P) we have

$$\begin{aligned}
\sum_{y \in Y_{x,i}} p_y &= \sum_{y \in Y_{x,i-1}} p_y + \sum_{y \in G_{i-1}} p_y \\
&\geq \frac{\varepsilon}{\varepsilon - \delta}(\mu_{i-1} - t_{x,i-1}) + \lambda_{i-1} \cdot \varphi_{i-1} \\
&\geq \frac{\varepsilon}{\varepsilon - \delta}(\mu_{i-1} - t_{x,i}) + \frac{\varepsilon}{\varepsilon - \delta}(\nu_{i-1} - \mu_{i-1}) \\
&= \frac{\varepsilon}{\varepsilon - \delta}(\mu_i - t_{x,i}).
\end{aligned}$$

If there is a job $z \in G_{i-1}$ with $r_z < t_{x,i-1} \leq \mu_{i-1}$, then this job belongs to $A_{i-1}$. During the construction of $G_i$, we chose $z$ to minimize $t_{z,i-1}$. Hence, $r_y \geq t_{y,i-1} \geq t_{z,i-1} = t_{x,i}$ holds for all $y \in G_{i-1}$. As $r_x \geq t_{x,i-1} > r_z \geq t_{z,i-1}$ holds, this shows (R). By construction $C_x^* \geq C_y^*$ holds for all $y \in G_{i-1}$ and, thus, $C_x^* \geq C_z^* \geq C_y^*$ also holds for all $y \in Y_{z,i-1}$; this is condition (C). For (P) consider

$$\begin{aligned}
\sum_{y \in Y_{x,i}} p_y &= \sum_{y \in Y_{z,i-1}} p_y + \sum_{y \in G_i} p_y \\
&\geq \frac{\varepsilon}{\varepsilon - \delta}(\mu_{i-1} - t_{z,i-1}) + \lambda_{i-1} \cdot \varphi_{i-1} \\
&= \frac{\varepsilon}{\varepsilon - \delta}(\mu_{i-1} - t_{x,i} + \nu_{i-1} - \mu_{i-1}) \\
&= \frac{\varepsilon}{\varepsilon - \delta}(\mu_i - t - x, i),
\end{aligned}$$

where the first inequality follows by induction on $z$, $Y_{z,i-1}$, and $t_{z,i-1}$, the second follows by construction, and the last equality holds because of $\nu_{i-1} = \mu_i$ by assumption.

**Showing $|X| \leq \frac{\alpha}{\beta}|J| + |\mathcal{I}|$.** By construction, we have that $\bigcup_{i \in \mathcal{I}_M} G_i = X$. As $|G_i| \leq \varphi_i$ holds for all $i \in \mathcal{I}_M$, the claim follows if $G_M = \emptyset$. The admission routine of the region algorithm guarantees that $F_M = \emptyset$. Hence, $G_M \neq \emptyset$ implies that $A_M \neq \emptyset$. However, (R), (P), and (C) also hold for any job $x \in A_M$ which in turn implies that such a job $x$ does not exist by Corollary 1. This concludes the proof of the lemma. □

*Proof of Theorem 4.* Clearly, $X \cup J$, the union of jobs only scheduled by OPT and the jobs the region algorithm admitted is a superset of the jobs that OPT completes on time. Lemma 6 shows that $|X| \leq \frac{\varepsilon}{\varepsilon-\delta}\frac{\alpha}{\beta}|J| + |\mathcal{I}|$. With Lemma 5 we can bound the cardinality of $\mathcal{I}$ by $2|J|$. Hence, we conclude

$$\text{OPT} \leq \left(\frac{\varepsilon}{\varepsilon-\delta}\frac{\alpha}{\beta} + 2\right)|J|.$$

□

## Conclusion

We close the major questions regarding online throughput maximization with commitment requirements on a single machine. It remains open whether randomization allows for improvements. Natural further questions concern generalizations such as weighted throughput maximization and parallel machine models. While strong lower bounds exist for handling weighted throughput with commitment [8], there remains a gap for the problem without commitment. For sufficiently large slack parameter, there is an online algorithm for parallel machines [2]. To the best of our knowledge, there is no algorithm for small slack.

## References


[1] K. Agrawal, J. Li, K. Lu, and B. Moseley. Scheduling parallelizable jobs online to maximize throughput. In *Proceedings of the Latin American Theoretical Informatics Symposium (LATIN)*, pages 755–776, 2018.

[2] Y. Azar, I. Kalp-Shaltiel, B. Lucier, I. Menache, J. Naor, and J. Yaniv. Truthful online scheduling with commitments. In *Proceedings of the ACM Symposium on Economics and Computations (EC)*, pages 715–732, 2015.

[3] N. Bansal, H.-L. Chan, and K. Pruhs. Competitive algorithms for due date scheduling. In *Proceedings of the International Colloquium on Automata, Languages and Programming (ICALP)*, pages 28–39, 2007.

[4] S. K. Baruah and J. R. Haritsa. Scheduling for overload in real-time systems. *IEEE Trans. Computers*, 46(9):1034–1039, 1997.

[5] S. K. Baruah, J. R. Haritsa, and N. Sharma. On-line scheduling to maximize task completions. In *Proceedings of the IEEE Real-Time Systems Symposium (RTSS)*, pages 228–236, 1994.

[6] S. K. Baruah, G. Koren, D. Mao, B. Mishra, A. Raghunathan, L. E. Rosier, D. E. Shasha, and F. Wang. On the competitiveness of on-line real-time task scheduling. *Real-Time Systems*, 4(2):125–144, 1992.

[7] R. Canetti and S. Irani. Bounding the power of preemption in randomized scheduling. *SIAM J. Comput.*, 27(4):993–1015, 1998.

[8] L. Chen, F. Eberle, N. Megow, K. Schewior, and C. Stein. A general framework for handling commitment in online throughput maximization. In *Proceedings of the Conference on Integer Programming and Combinatorial Optimization (IPCO)*, pages 141–154, 2019.

[9] B. DasGupta and M. A. Palis. Online real-time preemptive scheduling of jobs with deadlines. In *Proceedings of the International Conference on Approximation Algorithms for Combinatorial Optimization Problems (APPROX)*, pages 96–107, 2000.

[10] A. D. Ferguson, P. Bodík, S. Kandula, E. Boutin, and R. Fonseca. Jockey: guaranteed job latency in data parallel clusters. In *Proceedings of the European Conference on Computer Systems (EuroSys)*, pages 99–112, 2012.

[11] J. A. Garay, J. Naor, B. Yener, and P. Zhao. On-line admission control and packet scheduling with interleaving. In *Proceedings of the IEEE International Conference on Computer Communications (INFOCOM)*, pages 94–103, 2002.

[12] M. H. Goldwasser. Patience is a virtue: The effect of slack on competitiveness for admission control. In *Proceedings of the ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 396–405, 1999.

[13] S. Im and B. Moseley. General profit scheduling and the power of migration on heterogeneous machines. In *Proceedings of the ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, pages 165–173, 2016.

[14] B. Kalyanasundaram and K. Pruhs. Maximizing job completions online. *J. Algorithms*, 49(1):63–85, 2003.

[15] G. Koren and D. E. Shasha. MOCA: A multiprocessor on-line competitive algorithm for real-time system scheduling. *Theor. Comput. Sci.*, 128(1–2):75–97, 1994.

[16] G. Koren and D. E. Shasha. $\mathrm{D}^{\mathrm{over}}$: An optimal on-line scheduling algorithm for overloaded uniprocessor real-time systems. *SIAM J. Comput.*, 24(2):318–339, 1995.

[17] R. Lipton. Online interval scheduling. In *Proceedings of the ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 302–311, 1994.

[18] B. Lucier, I. Menache, J. Naor, and J. Yaniv. Efficient online scheduling for deadline-sensitive jobs: extended abstract. In *Proceedings of the ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, pages 305–314, 2013.

[19] K. Pruhs and C. Stein. How to schedule when you have to buy your energy. In *Proceedings of the International Conference on Approximation Algorithms for Combinatorial Optimization Problems (APROX)*, pages 352–365, 2010.

[20] C. Schwiegelshohn and U. Schwiegelshohn. The power of migration for online slack scheduling. In *Proceedings of the European Symposium of Algorithms (ESA)*, volume 57, pages 75:1–75:17, 2016.

[21] G. J. Woeginger. On-line scheduling of jobs with fixed start and end times. *Theor. Comput. Sci.*, 130(1):5–16, 1994.